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New symmetries and their Lie algebra properties for the Burgers equations

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Abstract. A general expression of one class of symmetries and their Lie algebra properties for the Burgers equation is presented.

1. Introduction

In this paper we discuss the symmetries and Lie algebra properties of the Burgers equation

$$u_t = u_{xx} + 2uu_x. \tag{1.1}$$

It has been shown (Stramp 1984, Li Yishen 1986) that this equation has one strong symmetry and two symmetries. The strong symmery is of the form

$$\Phi = \mathrm{D} + u + u_x\mathrm{D}^{-1} \tag{1.2}$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}x$; $\mathrm{D}^{-1} \cdot \mathrm{D} = \mathrm{D}\mathrm{D}^{-1} = \mathrm{I}$. The two symmetries are

$$K_0 = u_x \tag{1.3}$$

$$\tau_0 = 2tu_x + 1. \tag{1.4}$$

In a recent work, Tian Chou (1987a) obtained another strong symmetry

$$\Psi = 2t\Phi + x + \mathrm{D}^{-1} \tag{1.5}$$

and two symmetries

$$\sigma_0 = u \exp(-\mathrm{D}^{-1}u) \tag{1.6}$$

$$\Sigma_0 = (1-u) \exp(-\mathrm{D}^{-1}u + x + t). \tag{1.7}$$

Tian Chou also proved the relation

$$(\Phi, \Psi) = \mathrm{I} \tag{1.8}$$

and constructed the Lie algebra of the four groups of symmetries

$$K_n = \Phi^n K_0 \tag{1.9}$$

$$\tau_n = \Phi^n \tau_0 \tag{1.10}$$

$$\sigma_n = \Psi^n \sigma_0 \tag{1.11}$$

$$\Sigma_n = \Psi^n \Sigma_0 \tag{1.12}$$

where $n = 0, 1, 2, \ldots$.

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It is known that Burgers equation (1.1) can be mapped into a linear diffusion equation $\phi_t = \phi_{xx}$ through the Hopf-Cole transformation $u = \phi_x/\phi$ (Levi *et al* 1983). The existence of such a linearisation allows one to understand the origin of the rich mathematical properties of the Burgers equation (Degasperis and Leon *et al* 1983, Bruschi and Ragnixo 1985). Using this linearisation, it is not difficult to see that the symmetries σ_0 and Σ_0 are special cases of the symmetries of the Burgers equation. In the following we will show that generalisations of the above results are possible and a new class of symmetries are found with (1.6) and (1.7) included naturally.

2. Basic notation and lemmas

We consider the evolution equation

$$u_t = K(x, t, u, u_x, \ldots) \tag{2.1}$$

which depends on spatial variable x and time t explicitly.

Definition. $\sigma(x, t, u, u_x, \ldots)$ is called a symmetry of (2.1) if it satisfies the linear equation

$$\mathrm{d}\sigma/\mathrm{d}t = K'[\sigma] \tag{2.2}$$

where $\mathrm{d}\sigma/\mathrm{d}t$ is the total derivative, u satisfies (2.1) and $K'[\sigma]$ is the derivative of K in the direction σ:

$$K'[\sigma] = (\partial/\partial\varepsilon)K(u + \varepsilon\sigma)|_{\varepsilon=0}. \tag{2.3}$$

This yields

$$(\Phi K)'[\sigma] = \Phi'[\sigma]K + \Phi K'[\sigma] \tag{2.4}$$

where Φ is an operator and K is a function.

Definition. The operator Φ is called a strong symmetry of (2.1) if it maps a symmetry to another symmetry, i.e. if σ is a symmetry of (2.1), then $\Phi\sigma$ is also a symmetry of (2.1).

Definition. The operator Φ is a hereditary symmetry if

$$\Phi'[\Phi a]b - \Phi'[\Phi b]a = \Phi\{\Phi'(a)b - \Phi(b)a\} \tag{2.5}$$

is valid for any functions a and b. For instance, Φ of (1.2) and Ψ of (1.5) are two hereditary symmetries of the Burgers equation (Tian Chou 1987a).

Lemma 2.1. σ is a symmetry of (2.1) if and only if

$$\partial\sigma/\partial t = [K, \sigma] \tag{2.6}$$

where $\partial\sigma/\partial t$ is the partial derivative of σ with respect to t and the Lie product is given by

$$[K, \sigma] = K'[\sigma] - \sigma'[K]. \tag{2.7}$$

Lemma 2.2. The operator Φ is a strong symmetry of (2.1) if Φ satisfies

$$\mathrm{d}\Phi/\mathrm{d}t = [K', \Phi] \tag{2.8}$$

($[K', \Phi] = K'\Phi - \Phi K'$), i.e.

$$\Phi'[K] + \partial\Phi/\partial t = [K', \Phi].$$

Lemma 2.3. If Φ is a hereditary symmetry and

$$\Phi'[\sigma_0]=[\sigma_0',\Phi]$$

then

$$\Phi'[\sigma_n]=[\sigma_n',\Phi]$$

where

$$\sigma_n=\Phi^n\sigma_0 \qquad n=0,1,2,\ldots.$$

It is not difficult to prove these lemmas (Oevel and Fokas 1984, Tian Chou 1987b). Now we introduce a new lemma.

Lemma 2.4. If Φ is a hereditary symmetry, then

$$\Phi^n(\Phi^m)'(a)b-\Phi^m(\Phi^n)'(b)a=(\Phi^m)'(\Phi^n a)b-(\Phi^n)'(\Phi^m b)a \tag{2.9}$$

$(m=1,2,\ldots;\ n=1,2,\ldots)$ is valid for any functions a and b.

Proof. Obviously, the lemma is true for $m=1$, i.e.

$$\Phi^n\Phi'(a)b-\Phi(\Phi^n)'(b)a=\Phi'(\Phi^n a)b-(\Phi^n)'(\Phi b)a. \tag{2.9$'$}$$

Assuming (2.9) is true for $m=k-1$, we then have

$$\begin{aligned}\Phi^n(\Phi^k)'(a)b-\Phi^k(\Phi^n)'(b)a&\\ &=\Phi^n\Phi'(a)\Phi^{k-1}b+\Phi\{\Phi^n(\Phi^{k-1})'(a)b-\Phi^{k-1}(\Phi^n)'(b)a\}\\ &=\Phi^n\Phi'(a)\Phi^{k-1}b+\Phi(\Phi^{k-1})'(\Phi^n a)b-\Phi(\Phi^n)'(\Phi^{k-1}b)a\\ &=(\Phi^k)'(\Phi^n a)b-(\Phi^n)'(\Phi^k b)a.\end{aligned}$$

3. New symmetries of the Burgers equation

Theorem 3.1. Let

$$\mu_0(\varepsilon)=(u-\varepsilon)\exp(-\mathrm{D}^{-1}u+\varepsilon x+\varepsilon^2 t) \tag{3.1}$$

where ε is an arbitrary constant in time and space, then

$$\mathrm{d}\mu_0(\varepsilon)/\mathrm{d}t=K_1'[\mu_0(\varepsilon)] \tag{3.2}$$

i.e. $\mu_0(\varepsilon)$ is the symmetry of the Burgers equation (1.1).

Proof. From (1.1) (1.3) and (1.9), $K_1=u_{xx}+2uu_x$, it follows that

$$K_1'=\mathrm{D}^2+2u\mathrm{D}+2u_x.$$

Therefore

$$K_1'[\mu_0(\varepsilon)]=(u_{xx}+uu_x+\varepsilon u_x+\varepsilon u^2+\varepsilon^2 u-u^3-\varepsilon^3)\exp(-\mathrm{D}^{-1}u+\varepsilon x+\varepsilon^2 t)=\mathrm{d}\mu_0(\varepsilon)/\mathrm{d}t.$$

Theorem (3.1) gives a class of symmetries of the Burgers equation parametrised by ε. Choosing ε to be 0 and 1, (3.1) yields (1.6) and (1.7), respectively.

It is useful to note that, for the strong symmetry Φ given in (1.2),

$$\Phi\mu_0(\varepsilon)=\varepsilon\mu_0(\varepsilon). \tag{3.3}$$

In the following we will consider symmetries generated by the strong symmetry Ψ:

$$\mu_n(\varepsilon)=\Psi^n\mu_0(\varepsilon) \qquad n=0,1,2,\ldots. \tag{3.4}$$

4. Preliminary theorems

Lemma 4.1. The following equation holds:

$$\Phi\mu_n(\varepsilon)=\varepsilon\mu_n(\varepsilon)+n\mu_{n-1}(\varepsilon) \qquad n=0,1,2,\ldots. \tag{4.1}$$

The proof is by induction on n (using (3.3) and (1.8)) and is elementary.

Theorem 4.1.

$$\Phi^m\mu_n(\varepsilon)=\sum_{r=0}^{m} C_m^r[n!/(n-r)!]\varepsilon^{m-r}\mu_{n-r}(\varepsilon) \qquad m=1,2,\ldots, \qquad n=0,1,2,\ldots \tag{4.2}$$

Proof. Using lemma 4.1, (4.2) is true for $m=1$. By induction on m we have

$$\begin{aligned}\Phi^k\mu_n(\varepsilon)&=\Phi\Phi^{k-1}\mu_n(\varepsilon)\\&=\sum_{r=0}^{k-1} C_{k-1}^r[n!/(n-r)!]\varepsilon^{k-r-1}\Phi\mu_{n-r}(\varepsilon)\\&=\sum_{r=0}^{k-1} C_{k-1}^r[n!/(n-r)!]\varepsilon^{k-r-1}[\varepsilon\mu_{n-r}(\varepsilon)+(n-r)\mu_{n-r-1}(\varepsilon)]\\&=\sum_{r=0}^{k} C_k^r[n!/(n-r)!]\varepsilon^{k-r}\mu_{n-r}(\varepsilon).\end{aligned}$$

Lemma 4.2.

$$[\mu_n(\varepsilon)',x+\mathrm{D}^{-1}]=0 \qquad n=0,1,2,\ldots. \tag{4.3}$$

The proof is by induction on n.

Lemma 4.3.

$$\Phi'[\mu_0(\varepsilon)]=[\mu_0(\varepsilon)',\Phi] \tag{4.4}$$

$$\Psi'[\mu_0(\varepsilon)]=[\mu_0(\varepsilon)',\Psi]. \tag{4.5}$$

Proof. Using (1.2) and (3.1)

$$\Phi'[\mu_0(\varepsilon)] = \exp(-\mathrm{D}^{-1}u + \varepsilon x + \varepsilon^2 t)\{u - \varepsilon + [u_x - (u-\varepsilon)^2]\mathrm{D}^{-1}\}$$
$$= [\mu_0(\varepsilon)', \Phi].$$

Furthermore, from (1.5) and (3.1) we have

$$\Psi'(\mu_0(\varepsilon)) = 2t\Phi'[\mu_0(\varepsilon)]$$
$$[\mu_0(\varepsilon)', \Psi] = 2t[\mu_0(\varepsilon)', \Phi] + [\mu_0(\varepsilon)', x + \mathrm{D}^{-1}].$$

Thus (4.5) is valid by (4.3) and (4.4).

Theorem 4.2.

$$\Phi'[\mu_n(\varepsilon)] = [\mu_n(\varepsilon)', \Phi] \tag{4.6}$$
$$\Psi'[\mu_n(\varepsilon)] = [\mu_n(\varepsilon)', \Psi] \qquad n = 0, 1, 2, \ldots. \tag{4.7}$$

Proof. Since Ψ is a hereditary symmetry, according to (3.4), (4.5) and lemma 2.3, (4.7) is established. To prove (4.6), we notice, using (1.5), that

$$\Psi'[\mu_n(\varepsilon)] = 2t\Phi'[\mu_n(\varepsilon)]$$
$$[\mu_n(\varepsilon)', \Psi] = 2t[\mu_n(\varepsilon)', \Phi] + [\mu_n(\varepsilon)', x + \mathrm{D}^{-1}].$$

Thus from (4.3) and (4.7), one proves (4.6).

5. Lie algebra of the symmetries of Burgers equation

Theorem 5.1.

$$[\mu_m(\varepsilon), \mu_n(\varepsilon')] = 0 \tag{5.1}$$
$$[K_m, \mu_n(\varepsilon)] = \sum_{r=0}^{m+1} C_{m+1}^r [n!/(n-r)!]\varepsilon^{m-r+1}\mu_{n-r}(\varepsilon) \tag{5.2}$$
$$[\tau_m, \mu_n(\varepsilon)] = \sum_{r=0}^{m} C_m^r [(n+1)!/(n-r+1)!]\varepsilon^{m-r}\mu_{n-r+1}(\varepsilon) \tag{5.3}$$
$$m = 0, 1, 2, \ldots \qquad n = 0, 1, 2, \ldots$$

where ε and ε' are constants independent of space and time.

Proof of (5.1). It is easy to check that

$$[\mu_0(\varepsilon), \mu_0(\varepsilon')] = 0 \qquad [\mu_0(\varepsilon), \mu_1(\varepsilon')] = 0.$$

Hence from (4.7), (5.1) is obvious.

Proof of (5.2). For $m = 0$ (5.2) has the form

$$[K_0, \mu_n(\varepsilon)] = \varepsilon\mu_n(\varepsilon) + n\mu_{n-1}(\varepsilon) = \Phi\mu_n(\varepsilon). \tag{5.2'}$$

The case of $n = 0$ gives, using (1.3) and (3.1),

$$[K_0, \mu_0(\varepsilon)] = K_0'(\mu_0(\varepsilon)) - \mu_0(\varepsilon)'(K_0)$$
$$= \varepsilon\mu_0(\varepsilon) = \Phi\mu_0(\varepsilon).$$

Inducting on n, we have

$$[K_0, \mu_k(\varepsilon)] = [K_0, \Psi\mu_{k-1}(\varepsilon)]$$
$$= [K_0, 2t\varepsilon\mu_{k-1}(\varepsilon) + 2t(k-1)\mu_{k-2}(\varepsilon) + (x + \mathrm{D}^{-1})\mu_{k-1}(\varepsilon)]$$
$$= \Phi\mu_k(\varepsilon).$$

Next, assuming (5.2) is established for $m = k-1$, one has (using (2.4) and (4.6))

$$[K_k, \mu_n(\varepsilon)] = K'_k[\mu_n(\varepsilon)] - \mu_n(\varepsilon)'[K_k]$$
$$= \Phi'[\mu_n(\varepsilon)]K_{k-1} + \Phi K'_{k-1}[\mu_m(\varepsilon)] - \mu_n(\varepsilon)'[K_k]$$
$$= \Phi[K_{k-1}, \mu_n(\varepsilon)] = \Phi^{k+1}\mu_n(\varepsilon).$$

Thus from (4.2), (5.2) is valid.

Proof of (5.3). For $m = 0$, (5.3) has the form

$$[\tau_0, \mu_n(\varepsilon)] = \mu_{n+1}(\varepsilon). \tag{5.3'}$$

The case of $n = 0$ gives, using (1.4) and (3.1),

$$[\tau_0, \mu_0(\varepsilon)] = \tau'_0[\mu_0(\varepsilon)] - \mu_0(\varepsilon)'[\tau_0]$$
$$= \Psi\mu_0(\varepsilon) = \mu_1(\varepsilon).$$

Inducting on n, we have (using (1.5) and (4.1))

$$[\tau_0, \mu_k(\varepsilon)] = [\tau_0, \Psi\mu_{k-1}(\varepsilon)]$$
$$= [\tau_0, 2t\varepsilon\mu_{k-1}(\varepsilon) + 2t(k-1)\mu_{k-2}(\varepsilon) + (x + \mathrm{D}^{-1})\mu_k(\varepsilon)]$$
$$= \Psi\mu_k(\varepsilon) = \mu_{k+1}(\varepsilon).$$

Next, assuming (5.3) is established for $m = k-1$, one has (using (2.4) and (4.6))

$$[\tau_k, \mu_n(\varepsilon)] = \tau'_k[\mu_n(\varepsilon)] - \mu_n(\varepsilon)'[\tau_k]$$
$$= = \Phi'[\mu_n(\varepsilon)]\tau_{k-1} + \Phi\tau'_{k-1}[\mu_n(\varepsilon)] - \mu_n(\varepsilon)'[\tau_k]$$
$$= \Phi[\tau_{k-1}, \mu_n(\varepsilon)] = \Phi^k\mu_{n+1}(\varepsilon).$$

Thus from (4.2), (5.3) is valid.

Theorem (5.1) gives the Lie algebra of the symmetries (3.4) of the Burgers equation with the symmetries found previously. The results of the symmetries of higher-order Burgers equations will be reported in the following paper.

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